A bit of model theory

Thm (Tarski – Vaught criterion) Suppose $\mathcal{N} \subseteq \mathcal{M}$ are L-structures with $\mathcal{N}$ a substructure of $\mathcal{M}$. Then $\mathcal{N}$ is an elementary substructure of $\mathcal{M}$ if and only if

for any L-formula $A[x, v_1, \ldots, v_n]$ and $a_1, \ldots, a_n \in N$,

$$(*) \begin{cases} \text{if there is } b \in M \text{ with } \mathcal{M} \models A[b, a_1, \ldots, a_n] \\ \text{then there is } b \in N \text{ with } \mathcal{M} \models A[b, a_1, \ldots, a_n]. \end{cases}$$

Proof Suppose first that $\mathcal{N} \preceq \mathcal{M}$ and that for some $b \in M$, $\mathcal{M} \models A[b, \bar{a}]$. Then $\mathcal{M} \models \exists x\, A[x, \bar{a}]$, whence also $\mathcal{N} \models \exists x\, A[x, \bar{a}]$. It follows that for some $b \in N$, $\mathcal{N} \models A[b, \bar{a}]$, and hence $\mathcal{M} \models A[b, \bar{a}]$ for some $b \in N$.

Conversely, suppose that for any formula $A[x, \bar{v}]$ and $a_1, \ldots, a_n \in N$, $(*)$ holds. By induction on formulas, we show that if $B[\bar{v}]$ is an L-formula and $a_1, \ldots, a_n \in N$, then

$$(**) \quad \mathcal{N} \models B[\bar{a}] \quad \text{if and only if} \quad \mathcal{M} \models B[\bar{a}].$$

Since $\mathcal{N}$ is a substructure of $\mathcal{M}$, we know this holds for all quantifier free formulas.

Also if the induction hypothesis holds for formulas $B[\bar{v}]$ and $C[\bar{w}]$, then it holds for $\neg B$, $B \wedge C$, $B \vee C$, $B \rightarrow C$, $B \leftrightarrow C$.

Now, suppose the induction hypothesis (**) holds for $B[x,\bar{v}]$ and consider $\exists x B[x,\bar{v}]$.

Then, if $\mathcal{M} \models \exists x B[x,\bar{a}]$, by (*) there is some $b \in N$ such that $\mathcal{M} \models B[b,\bar{a}]$. Applying (**) to B, we have $\mathcal{N} \models B[b,\bar{a}]$ and so $\mathcal{N} \models \exists x B[x,\bar{a}]$.

Conversely, if $\mathcal{N} \models \exists x B[x,\bar{a}]$, then for some $b \in N \subseteq M$, $\mathcal{N} \models B[b,\bar{a}]$. By the induction hypothesis applied to B, also $\mathcal{M} \models B[b,\bar{a}]$, i.e., $\mathcal{M} \models \exists x B[x,\bar{a}]$.

Finally, note that for any formula B,

$\mathcal{M} \models \forall x B$ iff $\mathcal{M} \models \neg\exists x \neg B$ and similarly

$\mathcal{N} \models \forall x B$ iff $\mathcal{N} \models \neg\exists x \neg B$, so

$\forall$ can be eliminated using $\neg, \exists$.

This finishes the induction. $\square$

<u>Theorem</u> (Löwenheim - Skolem)

Let L be a language with only countably many symbols. Suppose $\mathcal{M}$ is an infinite L-structure and $X \subseteq M$ is a countable subset. Then there is an elementary substructure $\mathcal{N} \preceq \mathcal{M}$ such that $X \subseteq N$ and N is countable.

<u>Proof</u> We begin by noticing that since the language is countable there are only countably many finite strings of symbols from L and hence only countably many formulas of L.

Fix an arbitrary element $a_0 \in M$.

For every L-formula $B[x, v_1, \ldots, v_n]$, we define a function

$$\psi_{B[x, v_1, \ldots, v_n]} : M^n \longrightarrow M$$

by letting

$$\psi_{B[x,v_1,\dots,v_n]}(a_1,\dots,a_n) = \begin{cases} a_0 & \text{if } \mathcal{M} \models \neg\exists x\, B[x,a_1,\dots,a_n] \\ b & \text{for some } b \in M \text{ s.t. } \mathcal{M} \models B[b,a_1,\dots,a_n] \\ & \text{otherwise} \end{cases}$$

Now, if $Y \subseteq M$ is a countable subset, let

$$Y' = Y \cup \left\{ \psi_{B[x,v_1,\dots,v_n]}(a_1,\dots,a_n) \;\middle|\; a_1,\dots,a_n \in Y \;\&\; B[x,v_1,\dots,v_n] \text{ is a formula} \right\}.$$

Also, set

$$Y'' = \left\{ t[a_1,\dots,a_n]^{\mathcal{M}} \;\middle|\; a_1,\dots,a_n \in Y' \;\&\; t[a_1,\dots,a_n] \text{ is an } L\text{-term} \right\}$$

Note that as v_1 is a term $t[v_1]$, $a_1 = t[a_1]^{\mathcal{M}} \in Y''$ for any $a_1 \in Y''$.

<u>Claim</u> Y'' is countable.

For since L is countable there are only countably many functions $\psi_{B[x,v_1,\dots,v_n]}$ and terms $t[v_1,\dots,v_n]$. Thus, applying the ψ_B to Y only gives us countably many new points and then applying the $t^{\mathcal{M}}$ to Y' again only adds countably many points.

<u>Claim</u> Y'' is the domain of a substructure.

We only need to notice that if $f \in L$ is an n-ary function symbol and $a_1, \ldots, a_n \in Y''$, then also $f^{\mathcal{M}}(a_1, \ldots, a_n) \in Y''$.

So suppose $a_1 = t[\bar{b}_1]^{\mathcal{M}}, \ldots, a_n = t_n[\bar{b}_n]^{\mathcal{M}}$, where $t_1, \ldots, t_n$ are L-terms, then

$$f^{\mathcal{M}}(a_1, \ldots, a_n) = f^{\mathcal{M}}(t_1[\bar{b}_1]^{\mathcal{M}}, \ldots, t_n[\bar{b}_n]^{\mathcal{M}})$$
$$= s[\bar{b}_1, \ldots, \bar{b}_n]^{\mathcal{M}} \in Y''$$

where $s[\bar{w}_1, \ldots, \bar{w}_n]$ is the term $f(t_1, \ldots, t_n)$.

Now, define inductively,

$$X \subseteq X'' = X_0 \subseteq X_0'' = X_1 \subseteq X_1'' = X_2 \subseteq \ldots$$

and note that each X_n is countable.

It follows that $N = \bigcup_n X_n$ is countable too.

<u>Claim</u> $\mathcal{N} = \langle N, \ldots \rangle$ is an elementary substructure of $\mathcal{M}$.

First, to see that N is the domain of a substructure, assume $a_1, \ldots, a_n \in N$ and f is an n-ary fct. symbol. Since $N = \bigcup_m X_m$, there is some m s.t. $a_1, \ldots, a_n \in X_m = X_{m-1}^{n}$. So since X_{m-1}^{n} is the domain of a substructure, also $f^{\mathcal{M}}(a_1, \ldots, a_n) \in X_{m-1}^{n} \subseteq N$. Thus, also N is the domain of a substructure.

To see $\mathcal{N}$ is an elementary substructure, suppose $B[x, v_1, \ldots, v_n]$ is any $\mathcal{L}$-formula and $a_1, \ldots, a_n \in N$.) Suppose also $\mathcal{M} \models \exists x\, B[x, a_1, \ldots, a_n]$. Then, in particular, $\mathcal{M} \models B[b, a_1, \ldots, a_n]$, where $b = \psi_{B[x, v_1, \ldots, v_n]}(a_1, \ldots, a_n)$ and as above $b \in N$. Thus $\mathcal{N} \models \exists x\, B[x, a_1, \ldots, a_n]$.

So, by Tarski-Vaught, $\mathcal{N} \preceq \mathcal{M}$. $\square$

Thm $(\mathbb{Q}, <) \equiv (\mathbb{R}, <)$. I.e., for any sentence A of the language $L = \{<\}$, we have
$$(\mathbb{Q}, <) \models A \iff (\mathbb{R}, <) \models A.$$

Proof We note first that any two countable, dense linear orders w/o endpoints are isomorphic.

Also, $(\mathbb{R}, <)$ has a countable elementary substructure $\mathcal{N} = (N, <) \preccurlyeq (\mathbb{R}, <)$ by Löwenheim-Skolem. So as $(\mathbb{R}, <)$ is a dense linear order w/o endpoints, so is $\mathcal{N}$.

Thus $(\mathbb{Q}, <) \cong \mathcal{N} \preccurlyeq (\mathbb{R}, <)$, whence
$$(\mathbb{Q}, <) \equiv \mathcal{N} \equiv (\mathbb{R}, <). \qquad \square$$